PHYSICS 415/515 - Fall Semester 2025 - ODU

Nuclear and Particle Physics - Problem Set 9 - Solution

Problem 1)

The two terms that can be explained via the Fermi gas model are the asymmetry term and the pairing term (the last 2). The asymmetry term can be understood by the fact that the Fermi momentum for a given species of nucleons (protons or neutrons) increases if the total number of that species of nucleons increases, since it is proportional to the third root of that number. This means in turn that the kinetic energy for the species with the larger number (Z or N) increases more rapidly than just that number, leading to excess energy (mass) if the two species numbers aren't balanced (see also Problem 2). The pairing term accounts for the fact that the Fermi momentum for a nucleon species with one more member added increases if AND ONLY IF the total number of nucleons of that species goes from even to odd – there is no "penalty" in terms of the Fermi gas model to add one more nucleon with opposite spin into the same "phase space volume" occupied by only one nucleon. Therefore, the total kinetic energy per nucleon (and hence the total mass of the nucleus) is the lowest if both Z and N are even, while it is the highest if both are odd.

Problem 2)

For the following, we use the following equation for the Fermi momentum (from the lecture or the book – for the latter, solve Eq. 17.3 for p_F):

$p_f^{p/n} = \hbar \sqrt[3]{\frac{3\pi^2 Z/N}{V}} = 55.7 \text{ MeV}/c \sqrt[3]{Z/N}$, where $V = \frac{4\pi}{3} R^3 = 1317$ fm^3.

a) $p_f^p = 251.4$ MeV/c; $p_f^n = 293.3$ MeV/c.

b) Following the provided equation, the average kinetic energy per nucleon is for protons is 20.2 MeV and for neutrons it's 27.5 MeV, which is significantly larger (see also Problem 1 – this is excess energy is responsible for the large "asymmetry" term in Uranium.

c) The answer is given in part d): while the average kinetic energy for each proton is 7.3 MeV lower, there is an additional positive potential energy for them due to the Coulomb repulsion.

d) The energy of a charged sphere is given by $U = \frac{e^2}{4\pi\varepsilon_0}\frac{3}{5}\frac{Z^2}{R} = \alpha\hbar c\frac{3}{5}\frac{Z^2}{R}$. Therefore, the average energy per proton is $\alpha\hbar c\frac{3}{5}\frac{Z}{R} = 11.7$ MeV, which is even more than the excess kinetic energy per neutron. The discrepancy is due to the details of how exactly the total energy of the nucleus changes if a neutron is converted to a proton (including the fact that the nucleus would go from even-even to odd-odd) – plus the fact that the Fermi gas model is only a crude approximation.

Problem 3) (XC)

Microscopic ab-initio models of light nuclei calculate the interaction between each pair of nucleons within those nuclei and attempt to (approximately) solve the full Schrödinger equation for the resulting Hamiltonian (the sum of all kinetic energies and all nucleon-nucleon potential energies, summed over all nucleon-nucleon pairs). On the other hand, the Shell model assumes that each nucleon sees a universal “average” potential due to the presence of all other nucleons (averaged over all individual nucleon positions). By describing this average potential with a phenomenological ansatz, this effective potential can be solved for each individual nucleon in form of a 1-particle Schrödinger Equation, which leads to a series of eigenstates. The full nucleus can then be approximated by a combination of protons / neutrons occupying the lowest Z / N eigenstates of this Hamiltonian.

Obviously, the latter approach is much simpler and less computationally demanding; by fitting the effective potential parameters to the observed nuclear masses, a good description for heavier nuclei can be found. However, the shell model does not describe the details of nuclear structure, e.g., short-range correlations and other fluctuations in the wave function. On the other hand, even approximate methods like Greens Function Monte Carlos, Variational Monte Carlo etc. become unwieldy and impossible to solve for heavier nuclei, making effective models like the shell model unavoidable.